

# Coalitional Auctions for Complex Projects: Centralized and Decentralized View

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## Abstract

To successfully complete a complex project, be it a construction of an airport or of a backbone IT system or crowd-sourced projects, agents (companies or individuals) must form a team (a coalition) having required competences and resources. A team can be formed either by the project issuer based on individual agents' offers (centralized formation, corresponding to the set-system auctions in the literature); or by the agents themselves (decentralized formation) bidding for a project as a consortium—in that case many feasible teams compete for the employment contract. In these models, we investigate rational strategies of the agents (what salary should they ask? with whom should they team up?) under different organizations of the market. We propose various concepts allowing to characterize the stability of the winning teams. We show that there may be no (rigorously) strongly winning coalition, but the weakly winning and the auction-winning coalitions are guaranteed to exist. In a general setting, with an oracle that decides whether a coalition is feasible, we show how to find winning coalitions with a polynomial number of calls to the oracle. We also determine the complexity of the problem in a special case in which a project is a set of independent tasks. Each task must be processed by a single agent, but processing speeds differ between agents and tasks.

**keywords:** game theory, cooperative game theory, coalition formation, equilibria, skill games, scheduling, co-opetition

## 1 Introduction

Modern projects are getting more complex [5, 33]. They are getting complicated, involved, intricate, and they consist of many varied interrelated parts. The successful completion of such complex projects requires coordinated operation of a number of highly-specialized people, often organized as teams of sub-contractors [2]. For instance, in the home-building industry, typically, 30 to 40 individual sub-contractors are involved in completing 100 to 150 separate activities within a single residential construction [32].

Indeed, distributing parts of the complex projects among multiple sub-contractors is common [8]. According to Edwards [14], the proportions of constructions employees employed by sub-contractors in years 1983–1998 UK has grown by 20%. Between 2008 and 2011 the number of people who work as freelancers in UK has increased by 12% [24], and in 2012 in Australia, 17.2% of the workforce were self-employed (with 8.5% as independent contractors) [1]. These are only the few examples of the growing tendency. We envision that with the proliferation of maturing crowd-sourcing and collaboration platforms, we will witness the further growth in this tendency as crowd-sourcing will become an attractive paradigm for hiring sub-contractors.

Nevertheless, it is not clear how to organize the market for the complex project issuer (in this paper referred to as client) and the teams of sub-contractors (in this paper referred to as agents). The interaction between the agents, applying for an employment in a project, and the client is described in the *hiring a team* problem [18, 17, 3, 30, 9, 10, 20]. In the hiring a team problem, there is a client issuing a project, and a set of agents having their private costs of participating in the project. The agents may have different sets of skills and only certain coalitions are able to complete the project on time. The project is put on auction—after the agents came up with their bids, which indicate the amount of money they require for participation, the client selects the cheapest feasible coalition, i.e., the team of agents that is able to complete the project on time, with the lowest total bid.

We generalize the original approach from the hiring a team problem by exploring two organizations of the market. The original approach corresponds to the *centralized setting*, where the agents communicate only with the client by issuing their bids. In this paper we also consider a decentralized setting, where the whole already-formed teams of agents can bid for a project. To the best of our knowledge this formulation of the problem is novel and leads to the new class of games. Additionally, we generalize the original approach by considering two types of compensation of the agents. In the *project salary model* (corresponding to the original approach) the agents are payed for their whole participation in the project (irrespective of the contributed effort); in the *hourly salary model* the agents can decide how to organize their work, and are payed for the time spent working in the project.

Although our main contribution lies in considering the decentralized setting, we also complement the literature on the centralized one [18, 17, 3, 30, 9, 10, 20]. To the best of our knowledge, the current literature focuses on designing truthful mechanisms in the hiring a team problem that encourage agents to ask for the salaries corresponding to their actual costs of participation in a project. The known truthful mechanisms, however, result in a significant overpayment of the project issuer [17, 3, 30, 9, 10, 20]. In contrast to this literature, we consider the mechanism in which the agents are payed their asking salaries, and nothing more. This mechanism, which corresponds to the first-price-auctions, is manipulable, but once the agents set their asking salaries no overpayment is required. Since the first-price-auctions are, in general, more prone to collusions [22], we believe this auction system is relevant in our case.

Finally, in contrast to but also complementing the previous works, we focus on the computational aspects of finding winning coalitions rather than on the designing truthful mechanism.

Throughout the paper we assume that we are given an oracle that for a given coalition of agents can determine whether this coalition is feasible, i.e., whether it can successfully complete the project. Further, given a budget for the project, and the costs of the agents, the oracle can find some feasible coalition and the cheapest feasible coalition. Our approach generalizes two models known in the literature. In the commodity auctions [23] the project can be seen as the set of items  $I = \{i_1, i_2, \dots, i_q\}$  and each agent owns a certain subset of the items. The coalition is feasible if the agents have together all the items from  $I$ . In the path auctions [26] we are given a graph  $G$  with two distinguished vertices: a source  $s$  and a target  $t$ . The agents correspond to the vertices in the graph. The coalition is feasible if the participating agents form a path from  $s$  to  $t$ . For the general case, we also point out that the algorithms for coalition formation [28] can be used in oracle to solve the subproblem of finding (cheapest/best) feasible coalitions.

Since we consider coalitions of agents with sufficient skills to complete the project, our model resembles cooperative skill games [6] and coalitional resource games [34]. These games, however, consider the stability of the grand coalition and interaction between its members; our approach, on the other hand, is to expose multiple coalitions' competition. Thus, we do not apply the typical cooperative game theory concepts [27], and instead model the cooperation and the competition of the agents as the non-cooperative games.

The contributions of this paper are as follows: (i) First we identify and formalize a new class of coalition games, which are the extensions of the games from the *hiring a team* problem. In the *centralized setting* (Section 3), where the agents communicate only with the client, (ii) we prove that a Strong Nash Equilibrium (SNE) always exists unless there is no feasible coalition. We show how to find SNE, and for the client—how to select the best coalition, with a polynomial number of calls to the oracle. In a *decentralized setting* (Section 4) (iii) we show two concepts of winning coalitions. We prove that a strongly winning coalition may not exist, but a weakly winning coalition is guaranteed to exist (provided there exists a feasible one). We show how to find weakly/strongly winning coalitions. In Section 5 (iv) we propose two mechanisms that the client can apply to find the winning team. We introduce the concept of an auction-winning coalition and show how to find one. By specifying an oracle, our results can be applied to two different problems known in the literature: the commodity auctions and the path auctions. In Section 6 we propose another way in which the general oracle can be replaced with a concrete scheduling model (it is a generalization of the model from the commodity auctions) and (v) determine the exact complexity of the new concrete problem.

We also include a discussion on the related solution concepts and conclude the paper by pointing many interesting open problems.

## 2 The Auction Model for Complex Projects

We consider a model in which a client (an issuer) submits a single complex project to be executed. The client has a certain valuation  $v$  of the project, that is the maximal price that she is able to pay for completing the project.

There is a set  $N = \{1, 2, \dots, n\}$  of  $n$  agents. For each agent  $i$  we define  $\phi_i^{\min} > 0$  to be the agent's *minimal salary* for which  $i$  is willing to work. This minimal salary may correspond to the agent's personal cost of participating in the project. The agent prefers to work for  $\phi_i^{\min}$  than not to work (and then to work for higher salary). The value  $\phi_i^{\min}$  is private to the agent—neither the issuer nor the other agents know  $\phi_i^{\min}$ . However, in order to analyze the behavior of the agents in our system, we assume that agents have some beliefs about the minimal salaries of other agents.

A subset of the agents' population  $N$  forms a coalition to be awarded the project<sup>1</sup>; the paper's core contribution is on how this process should be organized.

A coalition  $\mathcal{C}$  is a triple  $\langle N_{\mathcal{C}}, \phi_{\mathcal{C}}, c_{\mathcal{C}} \rangle$  consisting of the set of participating agents  $N_{\mathcal{C}} \subseteq N$ , a salary function  $\phi_{\mathcal{C}} : N_{\mathcal{C}} \rightarrow \mathbb{N}$  assigning salaries to member agents, and the total cost of the coalition  $c_{\mathcal{C}} \in \mathbb{N}$ —the total amount of money earned by the participants of  $\mathcal{C}$ . Salaries are discrete to represent some minimal reasonable changes in the agents' compensation (not only money is discrete, but also it is common in real-world auctions to specify a minimal difference between two successive bids). However, to get some computational results in some, clearly marked, places we assumed that the salaries can be rational numbers.

The same coalition may organize the work of its members on the project in various ways. Each such a way may require different amount of effort from different participants. To capture this property we introduce a notion of a *schedule*,  $\sigma_{\mathcal{C}} : N_{\mathcal{C}} \rightarrow \mathbb{N}$ , that assigns to each member of a coalition the amount of time this agent needs to spend on the project. Of course there may exist many schedules for a single coalition. We will expand the discussion on the notion of schedule in Section 6.

We consider two models of agents' compensation. Let  $\phi_{\mathcal{C}}^{\text{tot}}(i)$  denote the total amount of money agent  $i$  gets in coalition  $\mathcal{C}$  (naturally,  $c_{\mathcal{C}} = \sum_{i \in N_{\mathcal{C}}} \phi_{\mathcal{C}}^{\text{tot}}(i)$ ). In the *project salary* model  $\phi_{\mathcal{C}}^{\text{tot}}(i)$  is equal to the salary of the agent  $\phi_{\mathcal{C}}(i)$  (and thus does not depend on the amount of work assigned to that agent). In the *hourly salary* model  $\phi_{\mathcal{C}}^{\text{tot}}(i)$  is equal to the product of the salary  $\phi_{\mathcal{C}}(i)$  and the time  $t_i$  that  $i$  spends on processing her part of the project ( $t_i$  is known from the schedule).

In the project salary model the agents are interested in earning as much money as possible. The hourly salary model represents a different environment in which agents perhaps work on many projects simultaneously; thus the agents are interested in having maximal salary per time unit (thus e.g. an agent prefers to work  $t_i = 1$  time unit with a salary  $\phi_i = 3$  to working  $t_i = 2$  time units with a salary  $\phi_i = 2$ ).

Different schedules might result in different completion times of the project. If the schedule results in a completion time that is satisfactory for the project issuer we say that

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<sup>1</sup>For the sake of the clarity of the presentation we refer to the teams of agents as to the coalitions, even if it sometimes abuses the definition of the coalition from the cooperative game theory.

the schedule is *feasible*. Of course for some coalitions, there might not exist a feasible schedule (e.g., if the coalition members are lacking certain critical skills). We assume that there is an oracle that can answer whether a given schedule is feasible or not. This very general setting can be specified by providing concrete models for the oracle. For instance, in Section 6 we show that by specifying an oracle, our results can be applied to two different problems known in the literature: the commodity auctions and the path auctions. In Section 6 we also show how to replace the general oracle with a new concrete scheduling model.

The coalition  $\mathcal{C}$  is *feasible* if there exist a feasible schedule such that: (i) the project budget is not exceeded ( $c_{\mathcal{C}} \leq v$ ); (ii) the cost  $c_{\mathcal{C}}$  of the coalition  $\mathcal{C}$  is consistent with the salaries  $\phi_{\mathcal{C}}$ . Specifically, in the project salary model  $c_{\mathcal{C}} = \sum_{i \in N_{\mathcal{C}}} \phi_{\mathcal{C}}(i)$ . In the hourly salary model  $c_{\mathcal{C}} = \sum_{i \in N_{\mathcal{C}}} t_i \phi_{\mathcal{C}}(i)$ , where  $t_i$  is the number of time units the agent  $i$  needs, according to the feasible schedule, to spend working on the project. Moreover, the asking salaries are no-lower than the minimal salaries,  $\phi_{\mathcal{C}}(i) \geq \phi_i^{\min}$ .

A coalition  $\mathcal{C}$  is *cheaper* than  $\mathcal{C}'$  if it has a strictly lower cost  $c_{\mathcal{C}} < c_{\mathcal{C}'}$  or if it has the same cost, but it is preferred by a deterministic tie-breaking rule  $\prec$ ,  $N_{\mathcal{C}} \prec N_{\mathcal{C}'}$  (for the sake of concreteness we assume that  $\prec$  is the lexicographic order in which a coalition is represented by a concatenation of the sorted list of the names of its members).

Throughout the paper we use the FIND FEASIBLE COALITION (FFC) and FIND CHEAPEST FEASIBLE COALITION (FCFC) problems. We reduce other problems to FFC and FCFC (we will also show that FCFC can be polynomially reduced to FFC).

**Problem 1** (FFC: Find Feasible Coalition). *An instance of FFC consists of a project (with a budget  $v$ ) and the set of the agents  $N$  with (known) minimal required salaries  $\phi_i^{\min}$ . The question is to find any feasible coalition or to claim there is no such.*

**Problem 2** (FCFC: Find Cheapest Feasible Coalition). *An instance is the same as in the FFC problem. The question is to find the cheapest feasible coalition or to claim there is no such.*

We use the general model as defined above in Sections 3, 4 and 5. Since the complexity of the above problems clearly depends on the underlying model for the oracle, we assume that our oracle can solve the FFC problem. This allows us to study a very general setting of the problem, abstracting from concrete notions of a schedule or a division of labor in a coalition.

To solve FFC, the oracle must know the underlying model, in particular the minimal salaries of agents and the maximal price of the project  $v$ ; we may also assume that the oracle is local to an agent—the oracle knows the minimal salary of this agent and agent’s beliefs about minimal salaries of others. Then, to get the exact computational results, we need to define in a compact form, e.g. which coalitions are able to complete the project. In Section 6 we consider a specific model of FFC in which the project is a set of independent, indivisible tasks that need to be completed before a certain deadline; and the agents have certain skills, i.e., speeds with which they process the tasks. We also discuss other specific models for the oracle that lead to the concrete models considered in the literature.

We consider two models of forming coalitions. First, in Section 3, we consider the *centralized formation*. This model is similar to the model from the *hiring a team* problem [18, 17, 3, 30, 9, 10, 20]. Agents submit their bids—(asking) salaries  $\phi_i$ —directly to the client (project issuer). The client chooses the members of the coalition that is awarded the project. Naturally, the client chooses the members so that the cheapest feasible coalition is formed. The members of the winning coalition are payed according to their asking salaries  $\phi_i$ . This is different from the literature on the set system auctions that considers different payment mechanisms that ensure the truthfulness of the agents [17, 3, 30, 9, 10, 20].

Second, in Sections 4 and 5, we consider the *decentralized formation* of the coalition. Agents communicate and are able to form coalitions by binding agreements. A coalition sends a bid—the total cost  $c_C$ —to the client; the bid represents the compensation the coalition expects to get for completing the whole project. The cheapest coalition  $C^*$  wins the project and is payed  $c_{C^*}$ ; then  $c_{C^*}$  is allotted to the members of the winning coalition according to the salary function  $\phi_{C^*}$ .

Our problem models various situations in which agents can form consortia to win a large project, including: crowdsourcing (e.g. amazon mechanical turk), online service sites, but also public tenders for large civil engineering projects (in which agents represent contractor companies specialized in a certain branch of construction or engineering).

### 3 Centralized Formation of Coalitions

In the centralized model we assume that the agents submit their asking salaries  $\phi_i$  directly to the client (the issuer of the project). The client, having the asking salaries of the agents, wants to form the cheapest feasible coalition (that is able to complete the project before the deadline). In this section, we first show that this problem reduces to FFC, the problem of finding a feasible coalition. Then, we analyze the optimal bidding strategies of agents.

**Proposition 1.** *The problem FCFC can be solved in time  $O((\log v + n)ffc)$ , where  $ffc$  is the complexity of the problem FFC.*

*Proof.* First, we solve FFC with binary search over  $v$  to find the lowest bid  $v^*$  for which there still exists a feasible coalition.

Next, we need to find the coalition bidding  $v^*$  that is preferred by the tie-breaking rule. We recall that  $C \prec C'$  if  $C$  precedes  $C'$  in the lexicographic order. We consider the agents in the increasing order of their names. For each agent  $i$  we decrease her salary by 1 ( $\phi_i^{\min} := \phi_i^{\min} - 1$ ) and solve FFC for  $v = v^* - 1$ . If there is one, this means that in the initial setting there exists a feasible coalition offering bid  $v^*$  and having agent  $i$  as a member. We store  $i$  as a member of the winning coalition. With the modified salary of  $i$  and an updated budget of  $v^* = v^* - 1$  we consider the next agent. Otherwise we reset the agent salary  $\phi_i^{\min}$  and the budget  $v^*$  to their previous values and consider the next agent.  $\square$

**Proposition 2.** *Having the asking salaries of the agents, the problem of finding the winning coalition can be solved in time  $O(fcfc)$ , where  $fcfc$  is the complexity of the problem FCFC.*

*Proof.* Solving the problem requires solving FCFC with the minimal salaries of the agents set to their asking salaries ( $\phi_i^{\min} = \phi_i$ ).  $\square$

The agents may behave strategically and manipulate their asking salaries to maximize their payoffs. This problem is a strategic game. An action of the agent  $i$  is her asking salary  $\phi_i \geq \phi_i^{\min}$ . The payoff of  $i$  is  $\phi_i$  if and only if  $i$  is a member of the cheapest feasible coalition; otherwise the payoff of  $i$  is 0.

Interestingly, in such setting, in the project salary model there exist sets which are stable against collaborative actions of the agents. We recall that a vector of the agents' actions is a Strong Nash Equilibrium (SNE, [4]) if no subset of the agents can change its actions so that all the deviating agents would obtain strictly better payoffs.

For each subset of the agents  $N' \subseteq N$ , by  $\mathcal{C}^*(N')$  we denote the cheapest feasible coalition using only the agents from  $N'$  (the coalition  $\mathcal{C}^*(N')$  does not exist if there is no feasible coalition consisting of the agents from  $N'$ ).

**Theorem 3.** *In the project salary model, if there exists a feasible coalition then there exists a Strong Nash Equilibrium. In every SNE, the set of the agents that get positive payoffs is the set of agents forming the cheapest feasible coalition,  $N_{\mathcal{C}^*(N)}$ .*

*Proof.* Let  $N^* = N_{\mathcal{C}^*(N)}$  be the set of the agents participating in the cheapest feasible coalition. We say that the action  $\phi_i$  of the agent  $i$  is *minimal* if and only if  $\phi_i = \phi_i^{\min}$ . We show how to construct the asking salaries  $\phi_i^*$  of the agents from  $N^*$  that, together with the minimal actions of the agents outside  $N^*$ , form the Strong Nash Equilibrium. A sketch of proof is as follows. We show the set of linear inequalities for the variables  $\phi_i, i \in N^*$ . Let us denote the maximal values of  $\phi_i$  which satisfy the inequalities as  $\phi_i^*$  (maximal in the sense that if we increase any value  $\phi_i^*$ , then the new values will not satisfy all the inequalities any more). We show that the actions  $\phi_i^*$  of the agents from  $N^*$ , together with the minimal actions of the agents outside of  $N^*$ , form an SNE and that the set of the solutions  $\phi_i^*$  that satisfy all the inequalities is nonempty.

The first inequality states that the values  $\phi_i$  must lead to a feasible solution:

$$\sum_{i \in N^*} \phi_i \leq v. \quad (1)$$

Next, as  $\mathcal{C}^*$  is the cheapest feasible coalition, for each feasible coalition  $\mathcal{C}'$  ( $N^* \neq N_{\mathcal{C}'}$ ) such that  $N^* \prec N_{\mathcal{C}'}$ ,  $\mathcal{C}^*$  must have (weakly) lower cost:

$$\sum_{i \in N^* \setminus N_{\mathcal{C}'}} \phi_i \leq \sum_{i \in N_{\mathcal{C}'} \setminus N^*} \phi_i^{\min}. \quad (2)$$

For  $\mathcal{C}'$  preferred over  $\mathcal{C}^*$  ( $N^* \neq N_{\mathcal{C}'}$  and  $N_{\mathcal{C}'} \prec N^*$ ),  $\mathcal{C}^*$  must have strongly lower cost:

$$\sum_{i \in N^* \setminus N_{\mathcal{C}'}} \phi_i < \sum_{i \in N_{\mathcal{C}'} \setminus N^*} \phi_i^{\min}. \quad (3)$$

First, if the values  $\phi_i^*$  satisfy the above inequalities and the agents outside of  $N^*$  play their minimal actions, then the agents from  $N^*$  will get positive payoffs. If they did not get the positive payoffs, it would mean that there exists a feasible cheaper coalition  $C'$ . However, the inequalities ensure that the agents from  $N^* \setminus N_{C'}$  induce the lower total cost than the total cost of the agents from  $N_{C'} \setminus N^*$ ; this ensures that agents  $N^*$  with actions  $\phi_i^*$  form a cheaper coalition than  $C'$ .

Next, we show that no set of agents  $N_{C'}$  can make a collaborative action  $\bar{\phi}$  after which the payoff of all agents from  $N_{C'}$  will be greater than previously. For the sake of contradiction let us assume that there exists such a set of agents  $N_{C'}$  and such an action  $\bar{\phi}$ . First we consider the case when the payoff of some agent  $i \notin N^*$  would change. This means that after  $\bar{\phi}$  there would be a new cheapest feasible coalition  $C'$ , where  $i \in N_{C'}$ . However, we know that the total cost of the agents from  $N^* \setminus N_{C'}$  is lower than the total cost of the agents from  $N_{C'} \setminus N^*$ . This means that  $C'$  cannot be cheaper than the coalition consisting of the agents from  $N^*$ . Finally, consider the case when only the payoffs of the agents from  $N^*$  change (and thus  $N_{C'} \subseteq N^*$ ). However, if the strict subset of  $N^*$  could form a feasible coalition, then  $C^*(N)$  would not be the cheapest. Thus,  $N_{C'} = N^*$ . This means that every agent from  $N^*$  must have played a higher action (and others must have not changed their actions). Since  $\phi_i^*$  were maximal, this means that after the action  $\bar{\phi}$  some inequality, for some feasible coalition  $C''$ , would not hold any more. Thus, we infer that  $C''$  is cheaper than  $C'$ .

To check that there always exists a solution, we see that the definition of  $N^*$  ensures that the values  $\phi_i^* = \phi_i^{\min}$  satisfy all inequalities.

Finally, by contradiction we prove the  $N^*$  is formed by the same agents as forming the cheapest coalition. Assume that the set of the agents that get positive payoffs in some SNE is  $N' \neq N^*$ . However, if the agents from  $(N^* \setminus N')$  play their minimal actions, then the coalition consisting of the agents from  $N^*$  would be cheaper than the coalition consisting of the agents from  $N'$ . Thus, the agents from  $(N^* \setminus N')$  can deviate, getting better payoffs. This completes the proof.  $\square$

Interestingly, there is no analogous result for hourly salary model.

**Proposition 4.** *In the hourly salary model there may not exist a Strong Nash Equilibrium even though there exists a feasible coalition.*

*Proof.* Let us consider the following instance. The budget is  $v = 49$ . There are 3 agents:  $a$ ,  $b$ , and  $c$ ; their minimal hourly salaries are  $\phi_a^{\min} = \phi_b^{\min} = \phi_c^{\min} = 1$ . All two-agent coalitions can complete the project: if  $a$  and  $b$  cooperate they can complete the project spending on it  $t_a = 10$  and  $t_b = 10$  time units, respectively; if  $a$  and  $c$  cooperate they must spend  $t_a = 22$  and  $t_c = 2$  time units; if  $b$  and  $c$  cooperate they must spend  $t_b = 2$  and  $t_c = 38$  time units.

For the sake of contradiction let us assume that there exists a Strong Nash Equilibrium. First, consider the case when the agents  $a$  and  $b$  get positive payoffs in SNE. By the budget constraint,  $\phi_b \leq 3$ . If  $\phi_b = 3$ , then  $\phi_a = 1$ . The total cost of  $\{a, b\}$  is 40. However,  $c$ , by playing  $\phi_c = 1$  can form a cheaper coalition  $\{a, c\}$  with the total cost 24. If  $\phi_b \leq 2$  and  $\phi_a = 1$ , then  $a$  has an incentive to play higher. If  $\phi_b \leq 2$  and  $\phi_a \geq 2$ , then  $b$  and  $c$  are



better off by playing a collaborative action with  $\phi_b = 3$  and  $\phi_c = 1$  — after such an action a coalition  $\{b, c\}$  is cheaper ( $c_{b,c} = 44$ ) than  $\{a, b\}$  ( $c_{a,b} \geq 50$ ) and  $\{a, c\}$  ( $c_{a,c} \geq 46$ ). Thus,  $a$  and  $b$  cannot both have positive payoffs in SNE.

Second, assume that the agents  $a$  and  $c$  get positive payoffs in SNE. The total cost of  $\{a, c\}$  is  $22\phi_a + 2\phi_c$ . In such case, if  $b$  plays  $\phi_a$  then the new coalition  $\{a, b\}$  with total cost  $10\phi_a + 10\phi_a$  forms a new cheapest coalition.

Finally consider the case when  $b$  and  $c$  get positive payoffs in SNE. This means that  $\phi_c = 1$ . But  $a$ , by playing 1 can form a coalition  $\{a, c\}$  with the total cost 24. This completes the proof.  $\square$

**Proposition 5.** *Checking whether a given vector of the asking salaries  $\langle \phi_i \rangle, i \in N$  is a Strong Nash Equilibrium can be solved in time  $O(fcfc)$ , where  $fcfc$  is the complexity of the problem FCFC.*

*Proof.* First, we find a winning coalition  $\mathcal{C}$  for  $\langle \phi_i \rangle$ . According to Proposition 2 we can do this by solving an instance of the FCFC problem (with  $\forall i : \phi_i^{\min} := \phi_i$ ). Next, we solve another instance  $I_2$  of the FCFC problem with the parameters set as follows. We set minimal salaries of the agents from  $N_{\mathcal{C}}$  to their asking salaries ( $\forall i \in N_{\mathcal{C}} \phi_i^{\min} := \phi_i$ ). The minimal salaries of the agents outside of  $N_{\mathcal{C}}$  are left unmodified. If the solution to  $I_2$  consists of the members of  $N_{\mathcal{C}}$  only, we claim that a vector  $\langle \phi_i \rangle, i \in N$  is a Strong Nash Equilibrium. Otherwise, it is not.  $\square$

The proof of Theorem 3 is constructive, however it requires considering all feasible coalitions, and so, leads to the potentially high complexity. On the other hand, if the salaries of the agents can be rational numbers, we can find the salary function in SNE by a polynomial reduction to the FCFC problem. This result is particularly interesting if the salaries of the agents have high granularity; rounding such a rational solution gives the integral solution which might not be exactly correct, but the error has a small magnitude.

**Proposition 6.** *In the project salary model, if the salaries of the agents are rational, then finding a Strong Nash Equilibrium can be solved in time  $O(n^3 \log(nv)fcfc)$ , where  $fcfc$  is the complexity of the problem FCFC.*

*Proof.* First, we solve a single instance of the FCFC problem to find  $N^* = N_{\mathcal{C}^*(N)}$ . Next, similarly as in the proof of Theorem 3, we introduce the variables  $\phi_i, i \in N^*$  and the inequalities (also the same as in the proof of Theorem 3). If we find the values  $\phi_i, i \in N^*$  satisfying all the inequalities, then the values  $\phi_i, i \in N^*$ , together with the minimal salaries of the agents outside of  $N^*$ , will form a Strong Nash Equilibrium.

The set of inequalities given in the proof of Theorem 3 is a linear program; there are, however, exponentially many constraints (a constraint for each possible coalition). We construct a separation oracle by a polynomial reduction to the FCFC problem. Since ellipsoid method [21] requires  $O(n^3 L)$  calls to the separation oracle [16] (where  $L$  is the size of the representation of the problem; here  $L = O(\log(nv))$ ), this allows us to solve the linear program in time  $O(n^3 \log(nv)fcfc)$ .

To check whether all the inequalities are satisfied, it is sufficient to solve FCFC with the following parameters. The minimal salaries of the agents from  $N^*$  are set to the values of the variables  $\phi_i$  ( $\forall_{i \in N^*} \phi_i^{\min} := \phi_i$ ). The minimal salaries of the agents outside of  $N^*$  are left unmodified. Let  $\mathcal{C}$  denote the solution of such instance of the FCFC problem. There exists a not-satisfied inequality if and only if  $N_{\mathcal{C}} \neq N^*$ . The not-satisfied inequality is the inequality that corresponds to the coalition  $\mathcal{C} \neq \mathcal{C}^*$ . This completes the proof.  $\square$

## 4 Decentralized Formation of Coalitions

Let us assume that the agents can communicate and agree their strategies. Consequently, they can form coalitions and bid for the project as consortiums. We show the concept of a (rigorously) strongly winning coalition, in which no subset of agents can successfully deviate. We show how to characterize (rigorously) strongly winning coalitions and how to reduce the problem of finding them to the FCFC problem. We show that the strongly winning coalitions may not exist, and so we introduce the concept of a weakly winning coalition. We prove that a weakly winning coalition always exists. We demonstrate how to reduce the problem of finding winning coalitions to the FCFC problem.

We model the behavior of the agents as a strategic game. Agent  $i$ 's action is a triple  $\langle N_{\mathcal{C}}, \phi_{\mathcal{C}}, b_{\mathcal{C}} \rangle$ . Intuitively, such an action means that the agent  $i$  decides to enter the coalition  $\mathcal{C} = \langle N_{\mathcal{C}}, \phi_{\mathcal{C}}, b_{\mathcal{C}} \rangle$ . The payoff of the agent is equal to  $\phi_{\mathcal{C}}(i)$  if (i)  $\mathcal{C}$  is feasible, (ii) each agent  $j \in N_{\mathcal{C}}$  agrees to participate in  $\mathcal{C}$  (i.e. they all play  $\mathcal{C}$ , and their payoffs are consistent with the bid of the coalition  $b_{\mathcal{C}}$ ), and (iii) there is no feasible cheaper coalition  $\mathcal{C}'$  such that all the agents from  $N_{\mathcal{C}'}$  agree to participate in  $\mathcal{C}'$ . Otherwise, the payoff of  $i$  is 0.

### 4.1 Strongly Winning Coalitions

In this game the payoffs depend on whether the others agree to cooperate, thus the Strong Nash Equilibrium (SNE) rather than the Nash Equilibrium [25] should be considered. In the following definition we propose an even more stable equilibrium concept than the SNE—the Rigorously Strong Nash Equilibrium (RSNE). The RSNE requires that no subset of agents can deviate in a way that each would get a payoff *at least as good* (instead of strictly better). Our approach is motivated by considering careful agents. In a SNE, the agents have no incentive to deviate if they get the same payoff; however they also have no incentive not to deviate. Yet, any deviation will result in a serious payoff loss for some agents (changing their payoffs from a positive  $\phi$  to zero). A careful agent will prefer not to be exposed to the possibility of such loss.

**Definition 1.** *The vector of actions  $\pi$  is a Rigorously Strong Nash Equilibrium (RSNE) if and only if there is no subset of the agents  $N_{\mathcal{C}}$  such that the agents from  $N_{\mathcal{C}}$  can make a collaborative action  $\mathcal{C}$  (a set of actions played by agents) after which the payoff of each agent  $i$  from  $N_{\mathcal{C}}$  would be at least equal to her payoff under  $\pi$  and the payoff of at least one agent  $i \in N$  would change.*

In the above definition the requirement that the payoff of at least one agent  $i \in N$  must change after the coalition deviates ensures that we treat as equivalent the coalitions with the same payoffs. For instance, assuming a system with three agents,  $a$ ,  $b$  and  $c$ , if the coalition  $\{a, b\}$  gets a positive payoff, it does not matter whether  $c$  plays  $\langle \{c\}, v + 1 \rangle$  or  $\langle \emptyset, v + 1 \rangle$ : in both cases all payoffs are the same.

Below we introduce additional definitions that help to characterize the Rigorously Strong Nash Equilibria in our game.

**Definition 2.** *A feasible coalition  $\mathcal{C}$  is explicitly endangered by a coalition  $\mathcal{C}'$  if (i)  $\mathcal{C}'$  is feasible, (ii)  $N_{\mathcal{C}} \cap N_{\mathcal{C}'} = \emptyset$  and (iii)  $\mathcal{C}'$  is cheaper than  $\mathcal{C}$ .*

*A feasible coalition  $\mathcal{C}$  is implicitly endangered by a coalition  $\mathcal{C}'$  if (i)  $\mathcal{C}'$  is feasible, (ii)  $N_{\mathcal{C}} \cap N_{\mathcal{C}'} \neq \emptyset$  and each agent from  $N_{\mathcal{C}} \cap N_{\mathcal{C}'}$  gets in  $\mathcal{C}'$  at least as good salary as in  $\mathcal{C}$ , and (iii) either  $N_{\mathcal{C}} \neq N_{\mathcal{C}'}$  or  $\phi_{\mathcal{C}} \neq \phi_{\mathcal{C}'}$ .*

If there are agents belonging to both coalitions ( $N_{\mathcal{C}} \cap N_{\mathcal{C}'} \neq \emptyset$ ), we do not consider the total cost of the alternative coalition  $\mathcal{C}'$ , as the decision whether  $\mathcal{C}'$  will be formed depends solely on the agents from  $N_{\mathcal{C}} \cap N_{\mathcal{C}'}$ : if they decide to form  $\mathcal{C}'$ ,  $\mathcal{C}$  will not be formed, thus the client won't be able to choose between  $\mathcal{C}$  and  $\mathcal{C}'$ .

Informally, a coalition is (rigorously) strongly winning if it constitutes a (rigorous) Strong Nash Equilibrium, i.e., the members will not deviate to other coalitions.

**Definition 3.** *The feasible coalition  $\mathcal{C}$  is rigorously strongly winning if and only if there is a Rigorously Strong Nash Equilibrium in which the agents from  $N_{\mathcal{C}}$  get positive payoffs  $\phi_{\mathcal{C}}$ .*

**Definition 4.** *The feasible coalition  $\mathcal{C}$  is strongly winning if and only if there is a Strong Nash Equilibrium in which the agents from  $N_{\mathcal{C}}$  get positive payoffs  $\phi_{\mathcal{C}}$ .*

The following theorem connects the intuitive notion of endangerment with the notion of a winning coalition.

**Theorem 7.** *The coalition  $\mathcal{C}$  is rigorously strongly winning if and only if  $\mathcal{C}$  is not explicitly nor implicitly endangered by any coalition.*

*Proof.*  $\Leftarrow$  Assume that there exists a rigorously strongly winning coalition  $\mathcal{C}$ ; thus there exists a Rigorously Strong Nash Equilibrium *RSNE* in which the agents from  $N_{\mathcal{C}}$  get positive payoffs. This implies that the agents from  $N_{\mathcal{C}}$  agree on the action  $\langle N_{\mathcal{C}}, \phi_{\mathcal{C}}, b_{\mathcal{C}} \rangle$ ; other agents ( $N \setminus N_{\mathcal{C}}$ ) have zero payoffs. For the sake of contradiction let us assume that there exists a feasible coalition  $\mathcal{C}'$  such that  $\mathcal{C}$  is explicitly or implicitly endangered by  $\mathcal{C}'$ .

If  $N_{\mathcal{C}} \cap N_{\mathcal{C}'}$  is empty ( $\mathcal{C}$  is explicitly endangered by  $\mathcal{C}'$ ), then  $N_{\mathcal{C}'}$  must be cheaper. This however contradicts the assumption that the agents from  $N_{\mathcal{C}}$  get positive payoffs.

Assume thus that  $N_{\mathcal{C}} \cap N_{\mathcal{C}'}$  is non-empty (i.e.,  $\mathcal{C}$  is implicitly endangered by  $\mathcal{C}'$ ). Consider the following collaborative action of agents  $(N \setminus N_{\mathcal{C}}) \cup N_{\mathcal{C}'}$ . All the agents from  $N_{\mathcal{C}'}$  make action  $\mathcal{C}'$ . Each agent  $i$  from  $N \setminus (N_{\mathcal{C}} \cup N_{\mathcal{C}'})$  makes an action  $\langle \{\}, \phi_{\emptyset} \rangle$ , where  $\phi_{\emptyset}$  is an empty function. We show that after playing this action no agent from  $(N \setminus N_{\mathcal{C}}) \cup N_{\mathcal{C}'}$  will get lower payoff and that some agents will get a strictly better payoff (which will contradict

the assumption that *RSNE* is a Rigorously Strong Nash Equilibrium). Clearly each agent from  $N \setminus (N_C \cup N_{C'})$  does not decrease her payoff (as previously it was equal to 0). Now, we show that the agents from  $N_{C'}$  will get at least the same payoff as before. Since we know that  $C$  is implicitly endangered by  $C'$  (and thus the agents from  $N_C \cap N_{C'}$  get in  $C'$  at least as good payoff as in  $C$ ) it is sufficient to show that the agents from  $N_{C'}$  will get positive payoffs. Indeed, there is no feasible coalition that includes some agents from  $N \setminus (N_C \cup N_{C'})$  (as these agents play  $\{\}$ ). Also, the agents from  $N_C \setminus N_{C'}$  do not agree on the collaborative action (they still play  $C$ ) and thus, cannot form a feasible coalition. Thus, after such change of played actions  $C'$  is the only feasible coalition that the members agreed on. Finally, we can show that at least one agent will get a strictly better payoff. Either  $N_C = N_{C'}$  (and since  $\phi_C \neq \phi_{C'}$ , some agent must get a different payoff) or  $N_C \neq N_{C'}$  (and the agents from  $N_{C'} \setminus N_C$  will get a positive payoff).

$\implies$  Assume that  $C$  is not explicitly nor implicitly endangered by any coalition. First, if the agents from  $N_C$  make the collaborative action  $C$ , then they will all get positive payoffs. Indeed, the agents in  $N_C$  could not get positive payoffs only if there would exist a cheaper feasible coalition  $C'$  such that  $N_C \cap N_{C'} = \emptyset$ . This would, however mean that  $C$  is explicitly endangered by  $C'$ . Next, we show that the state in which the agents from  $N_C$  make the collective decision  $C$  and the other agents play arbitrary actions is RSNE. For the sake of contradiction let us assume that there exists a subset of agents  $N_{C'}$  which can make a collaborative action  $C'$  after which the payoff of everyone from  $N_{C'}$  would be at least equal to her payoff in  $C$ . This would, however mean that  $C$  is either implicitly or explicitly endangered by  $C'$ . This completes the proof.  $\square$

The result in Theorem 8 stated for Rigorously Strong Nash Equilibria transfers to Strong Nash Equilibria with a slight modification of the model. It is sufficient to assume that the payoff of an agent who plays an empty coalition is slightly higher than in case of playing non-empty losing coalition. In other words, this modification associates some small costs of preparing a bid by agents. Hereinafter, whenever we mention a strictly winning coalition we assume that the agents have small but positive costs of preparing the offer. To state the result for Strong Nash Equilibria we also need to use the definition of a coalition  $C$  being *strictly implicitly endangered* by  $C'$ . This definition differs from being implicitly endangered only by the fact that we do not require the agents from  $N_C \cap N_{C'}$  to have at least as good payoffs, but strictly better payoffs in  $C'$  than in  $C$ .

**Theorem 8.** *If there are small but positive costs of preparing the offer by the agents then the coalition  $C$  is strongly winning if and only if  $C$  is not explicitly nor strictly implicitly endangered by any coalition.*

*Proof.* The proof is analogous to the proof of Theorem 7.  $\square$

Theorems 7 and 8 give us better understanding of the concept of Rigorously Strong Nash Equilibrium (and Strong Nash Equilibrium) in our model. They also lead to a simple brute-force algorithm for checking whether the coalition can be a part of some RSNE. Below,

we provide the analysis that allows to characterize RSNE in the project salary model even more precisely.

**Theorem 9.** *In the project salary model the set of agents participating in a rigorously strongly winning coalition is the same as the set of agents participating in the cheapest feasible coalition.*

*Proof.* Let  $\mathcal{C}$  denote the cheapest feasible coalition. We show that for any other coalition  $\mathcal{C}'$ , such that  $N_{\mathcal{C}} \neq N_{\mathcal{C}'}$ ,  $\mathcal{C}'$  cannot be rigorously strongly winning. For the sake of contradiction let us assume that  $\mathcal{C}'$  is rigorously strongly winning. Let  $N_{\cap} = N_{\mathcal{C}} \cap N_{\mathcal{C}'}$ . Since  $\mathcal{C}$  is the cheapest, the sum of salaries of the agents from  $N_{\mathcal{C}} \setminus N_{\cap}$  in  $\mathcal{C}$  is lower or equal to the sum of salaries of the agents from  $N_{\mathcal{C}'} \setminus N_{\cap}$  in  $\mathcal{C}'$ . Consider a coalition  $\mathcal{C}''$  consisting of the set of agents  $N_{\mathcal{C}}$  and the following salary function. The salary of each agent from  $N_{\mathcal{C}} \setminus N_{\cap}$  is the same as in  $\mathcal{C}$  and the salary of each agent from  $N_{\cap}$  is the same as in  $\mathcal{C}'$ . Since the bid  $c_{\mathcal{C}'}$  of  $\mathcal{C}'$  was below  $v$ , the bid of  $\mathcal{C}''$  is also below  $v$ . Thus,  $\mathcal{C}''$  is feasible. Also,  $\mathcal{C}'$  is implicitly endangered by  $\mathcal{C}''$ , which leads to contradiction and completes the proof.  $\square$

Interestingly, there is no analogous result for the hourly salary model.

**Proposition 10.** *In the hourly salary model the set of the agents participating in a rigorously strongly winning coalition may not be the same as the set of the agents participating in the cheapest feasible coalition.*

*Proof.* Let us consider 3 agents  $a$ ,  $b$ , and  $c$  with equal minimal salaries  $\phi_a^{\min} = \phi_b^{\min} = \phi_c^{\min} = 1$ . Agents  $a$  and  $b$  can complete the project spending  $t_a = 10$  and  $t_b = 10$  time units on it. Also, the agents  $b$  and  $c$  can complete the project spending  $t_b = 4$  and  $t_c = 20$  time units on it. The maximal budget is  $v = 28$ . The cheapest coalition is  $\{a, b\}$  with salaries  $\phi_a = \phi_b = 1$  and bid equal to 20. However, the coalition  $\{b, c\}$  with salaries  $\phi_b = 2$  and  $\phi_c = 1$  is rigorously strongly winning.  $\square$

**Proposition 11.** *In the project salary model the bid of a strongly winning coalition is equal to the maximal allowed price  $v$ .*

*Proof.* Let  $\mathcal{C}$  be a strongly winning coalition. If  $b_{\mathcal{C}} < v$  we could increase the salaries of some participating agents. The resulting coalition would implicitly endanger  $\mathcal{C}$ .  $\square$

Proposition 11 is a consequence of the fact that the coalition  $\mathcal{C}$  can be endangered by a coalition  $\mathcal{C}'$  having the same set of the participants but different salaries. An alternative characterization disallows such behavior; we do not explore this path further in our paper.

Theorem 9 and Proposition 11 show that the problem of finding a strongly winning coalition collapses to the problem of finding a feasible coalition. The problem thus becomes an optimization problem; the strategic behavior of agents does not have an influence on this procedure.

**Proposition 12.** *Checking whether a coalition is rigorously strongly winning can be solved in time  $O(n^2 \cdot \text{fcfc})$ , where  $\text{fcfc}$  is the complexity of the problem FCFC.*

*Proof.* Let us assume that we want to check whether the coalition  $\mathcal{C}$  is rigorously strongly winning. First, we check whether we can increase the salary of any agent so that the coalition would still be feasible. If we can,  $\mathcal{C}$  is not rigorously strongly winning. Otherwise, we solve FCFC for the set of agents  $N \setminus N_{\mathcal{C}}$ . If there exists a non-empty solution  $\mathcal{C}'$  with the cost  $c_{\mathcal{C}'} < c_{\mathcal{C}}$  or such that  $c_{\mathcal{C}'} = c_{\mathcal{C}}$  and  $\mathcal{C}' \prec \mathcal{C}$ , this means that  $\mathcal{C}$  is explicitly endangered by  $\mathcal{C}'$ , and thus is not rigorously strongly winning. Otherwise,  $\mathcal{C}$  is not explicitly endangered by any coalition.

Next, we check whether  $\mathcal{C}$  is implicitly endangered by some coalition  $\mathcal{C}'$ . We change the names of the agents so that the agents from  $N_{\mathcal{C}}$  were the first  $\|N_{\mathcal{C}}\|$  agents in the lexicographic order. Now, for each agent  $i$  from  $N_{\mathcal{C}}$  we do the following procedure. We solve FCFC for the set of agents  $N \setminus \{i\}$ , for the minimal salaries of the agents from  $N_{\mathcal{C}}$  changed to their salaries in  $\mathcal{C}$ , and for the budget  $v$  set to  $c_{\mathcal{C}}$ . If there exists a feasible  $\mathcal{C}'$  to FCFC such that the set  $N_{\mathcal{C}'}$  overlaps with  $N_{\mathcal{C}}$  (overlapping can be tested in time  $O(n)$ ), then  $\mathcal{C}$  is implicitly endangered by  $\mathcal{C}'$ . We already know that there is no non-overlapping coalition with the cost lower than  $c_{\mathcal{C}}$ . Thus, if for no agent  $i$  from  $N_{\mathcal{C}}$  we find such implicitly endangering coalition, this means that there is no feasible coalition  $\mathcal{C}'$  such that  $N_{\mathcal{C}} \cap N_{\mathcal{C}'} \neq \emptyset$ . Thus, in such case we conclude that  $\mathcal{C}$  is rigorously strongly winning.  $\square$

**Proposition 13.** *In the project salary model, if the salaries of the agents can be rational numbers, finding a rigorously strongly winning coalition can be solved in time  $O(n^5 \log(nv)fcfc)$ , where  $fcfc$  is the complexity of the problem FCFC.*

*Proof.* First we solve FCFC to find the cheapest coalition  $\mathcal{C}$ . We know that the set of the agents participating in a rigorously strongly winning coalition is  $N_{\mathcal{C}}$  (Theorem 9) and the total cost of such a coalition is  $v$  (Proposition 11). We only need to find the salary function of such a coalition. For every agent  $i$  from  $N_{\mathcal{C}}$ , we introduce a variable  $\phi_{\mathcal{C}}(i)$ . We will show the linear program for the variables  $\phi_{\mathcal{C}}(i)$ , to which the solution is a rigorously strongly winning coalition. At the same time we will show how to implement the separation oracle for the linear program.

First equality states that the salaries of the agents satisfy the feasibility constraint:

$$\sum_{i \in N_{\mathcal{C}}} \phi_{\mathcal{C}}(i) = v \quad (4)$$

Next two inequalities model explicit endangerment. For each coalition  $\mathcal{C}'$ , such that  $N_{\mathcal{C}} \cap N_{\mathcal{C}'} = \emptyset$  and  $\mathcal{C}' \prec \mathcal{C}$ :

$$\sum_{i \in N_{\mathcal{C}}} \phi_{\mathcal{C}}(i) < \sum_{i \in N_{\mathcal{C}'}} \phi_i^{\min}. \quad (5)$$

For each coalition  $\mathcal{C}'$ , such that  $N_{\mathcal{C}} \cap N_{\mathcal{C}'} = \emptyset$  and  $\mathcal{C} \prec \mathcal{C}'$ :

$$\sum_{i \in N_{\mathcal{C}}} \phi_{\mathcal{C}}(i) \leq \sum_{i \in N_{\mathcal{C}'}} \phi_i^{\min}. \quad (6)$$

Note that we can check the above two inequalities by solving FCFC problem for the set of agents  $N \setminus N_C$ . If the resulting coalition  $C'$  is cheaper than  $C$ , this means that the inequality constraint for  $C'$  was violated. Otherwise, all the above inequalities are satisfied.

Last, for each coalition  $C'$ , such that  $N_C \cap N_{C'} \neq \emptyset$  and  $N_C \neq N_{C'}$  we introduce the inequality modeling implicit endangerment:

$$\sum_{i \in N_C \setminus N_{C'}} \phi_C(i) + \sum_{i \in N_{C'} \setminus N_C} \phi_i^{\min} > v. \quad (7)$$

We can check this inequality in the same way as we checked whether the coalition was implicitly endangered in the proof of Proposition 12: by swapping the names of the agents, for each  $i \in N_C$  solving FCFC for the set of agents  $N \setminus \{i\}$ , and checking the overlapping of the appropriate sets. The whole procedure requires the time  $O(n^2 \cdot fcfc)$ .

As the result, we showed the reduction of the problem of finding a rigorously strongly winning coalition to the linear program with  $n$  variables and a separation oracle running in time  $O(n^2 \cdot fcfc)$ .  $\square$

It is desired to know Rigorously Strong Nash Equilibrium provided they exist. However a RSNE (and even a Strong Nash Equilibrium) may not exist in some instances.

**Proposition 14.** *Both in the project salary and in the hourly salary model, there may not exist a strongly winning coalition even though there exists a feasible coalition.*

*Proof.* Consider a project with budget  $v = 5$ ; and three identical agents  $a, b, c$  with minimal salaries  $\phi_i^{\min} = 2$  (in the hourly salary model, assume that each agent spends exactly 1 time unit on the project); a coalition of any two agents is feasible (able to complete the project on time and within the budget).

For the sake of contradiction assume there exists a coalition  $C$  that gets positive payoffs. Without loss of generality we assume that  $N_C = \{a, b\}$ . At least one of the agents, let us say  $a$  has to get salary equal to 2. However, the agents  $a$  and  $c$ , with the salaries equal to 3 and 2 respectively, can form a feasible coalition in which both  $a$  and  $c$  get better payoffs (note that we use here the fact that the payoffs are discrete).  $\square$

## 4.2 Weakly Winning Coalitions

We are not fully satisfied with the example from Proposition 14. Indeed the coalition  $\{a, c\}$  can profit by deviating, but  $a$  should not be willing to deviate. The reason is that  $\{a, c\}$  is not stable by its own, and the coalition  $\{b, c\}$  can be successfully deviate it. In the above example no coalition will be formed even though intuitively we feel that there are coalitions that would agree to work. Thus, we propose a weaker notion.

**Definition 5.** *A feasible coalition  $C$  is weakly winning if it is not explicitly endangered by any coalition and for each feasible coalition  $C'$  such that  $C$  is implicitly endangered by  $C'$ , there exists a feasible coalition  $C''$  such that  $C'$  is explicitly or implicitly endangered by  $C''$ .*

**Proposition 15.** *There exists a weakly winning coalition if and only if there exists a feasible coalition.*

*Proof.* Consider a feasible coalition  $\mathcal{C}$  that is not explicitly endangered (such a coalition exists provided there exists a feasible coalition). Let  $\mathcal{E}$  denote a set of feasible coalitions implicitly endangering  $\mathcal{C}$ . If  $\mathcal{E} = \emptyset$ ,  $\mathcal{C}$  is strongly winning and, thus also, weakly winning. If there exists  $\mathcal{C}' \in \mathcal{E}$  such that  $\mathcal{C}'$  is not (implicitly or explicitly) endangered by any feasible coalition, then  $\mathcal{C}'$  is strongly winning (and, thus also, weakly winning). Otherwise,  $\mathcal{C}$  is weakly winning.

If there is no feasible coalition then there is no weakly winning coalition.  $\square$

**Proposition 16.** *In the project salary model, if the salaries of the agents can be rational numbers, the problem of finding a weakly winning coalition can be solved in time  $O(n^5 \log(nv)fcfc)$ , where  $fcfc$  is the complexity of the problem FCFC.*

*Proof.* First, we look for a rigorously strongly winning coalition. If there is one, it is also weakly winning, and so the procedure is complete. If there is no rigorously strongly winning coalition it is sufficient to find a coalition that is not explicitly endangered by any other coalition. We can do this by solving a single instance of the FCFC problem.  $\square$

**Proposition 17.** *In the project salary model, if the salaries of the agent can be rational numbers, the problem of checking whether a coalition  $\mathcal{C}'$  is weakly winning coalition can be solved in time  $O(n^5 \log(nv)fcfc)$ , where  $fcfc$  is the complexity of the problem FCFC.*

*Proof.* We first check whether the coalition is explicitly endangered by any other coalition. We can do this by solving a single instance of the FCFC problem for the set of agents  $N \setminus N_{\mathcal{C}'}$ .

Next we look for a rigorously strongly winning coalition that endangers  $\mathcal{C}'$ . We do this in the same way as in the proof of Proposition 13. The only difference is that we additionally introduce the following inequalities. We assume the same notation as in the proof of Proposition 13. For each  $i \in N_{\mathcal{C}} \cap N_{\mathcal{C}'}$  we require:  $\phi_{\mathcal{C}}(i) \geq \phi_{\mathcal{C}'}(i)$ .  $\square$

### 4.3 Other Solution Concepts

In this section we give a brief overview of other solution concepts that can be applied to describe winning coalitions in our game. Most of these solution concepts have their drawbacks, and they do not allow to determine winning coalitions. On the other hand we point out the two ideas that, we believe, are interesting for further analysis. The first idea is to apply the concept of the Coalitional Farsighted Conservative Stable Set [12] to our setting. The second is to apply the concepts inspired by the graph interpretations. These two solution concepts are, however, more involved, and so, we believe that the our definition of the weakly winning coalition is the natural simplification, and the first step to understand the complexity of the agents' interactions.

In the subsequent subsections we present the discussion on the application of different solution concepts to our model.



### 4.3.1 Cooperative Game Theory Approach

It may seem that our solution concepts are closely related to the solution concepts from the cooperative game theory. For instance the definition of Rigorously Strong Nash Equilibrium is close in spirit to the concept of core from the cooperative games. However, there are some substantial differences. In cooperative game theory it is commonly assumed that the value of the coalition depends only on this coalition. The following example shows that this is not the case in our problem.

**Example 1.** Consider 2 agents  $a$  and  $b$  with the minimal salaries  $\phi_a^{\min} = 1$  and  $\phi_b^{\min} = 2$ . The maximal budget of the issuer is  $v = 2$ . Consider two coalitions formed by single agents  $C_1 = \{a\}$ , and  $C_2 = \{b\}$ . Let us assume that  $C_2$  is feasible. The value of  $C_2$  depends on whether the agent  $C_1$  is feasible or not.

The above example encourages to consider our problem as a cooperative game with externalities. However, in such games the value of the coalitions only depend on the partition of the agents into coalitions. In our case however the whole coalitions are strategic, and their value depends on the action (the bid) of the coalitions. We provide a detailed discussion on the application of the selected concepts from cooperative game theory in the two following subsections.

### 4.3.2 The Core

Although the notion of *the core* is initially known from the cooperative game theory, there is a natural generalization to strategic games. In this generalization we say that the coalition  $C$  with the payoff function  $\phi$  is in the core if and only if there is no feasible coalition  $C'$  with the payoff function  $\phi'$  such that every agent in  $C'$  gets according to  $\phi'$  better payoff than according to  $\phi$ .

Although, in cooperative game theory we use a simplified model in which the feasibility means just that the total payoff of the agents do not exceed the value of the coalition (i.e., the bid of the coalition, in our approach), we may use the more demanding notion of feasibility from our model. As the result, the coalition  $C$  is in the core if and only if it is not implicitly endangered by any other coalition.

Intuitively, the notion of the core in our game is missing an important element. Indeed, the coalition  $C$  might be in the core even though some other coalition  $C'$ , disjoint with  $C$  can offer a better price, and consequently win the auction and be awarded the project.

### 4.3.3 The (Farsighted) Stability

Another notion known from the cooperative game theory that is worth considering is the von Neumann-Morgenstern stable set. The stable set is the set of all payoff vectors such that (i) no payoff vector in the stable set is dominated by another vector in the set, and (ii) all payoff vectors outside the set are dominated by at least one vector in the set.

In the light of our previous example from Proposition 14, it is even more appealing to consider the farsighted von Neumann-Morgenstern stable set [11]. The farsighted coalition

is more deliberative, it considers that if it makes a deviation, the second coalition might react as a consequence of the first coalition's action, next the third coalition might react, and so on without the limit. In the original formulation the agents are considered to be optimistic—they are willing to deviate if the deviation starts some sequence of deviations that would lead to a better outcome.

In our game the vN-M stable set, and the farsighted vN-M stable set might be empty.

**Example 2.** *Consider the example from Proposition 14. There is a project with the budget  $v = 5$ ; and three identical agents  $a, b, c$  with minimal salaries  $\phi_i^{\min} > 2$ . Every coalition formed by any two agents is feasible. For the sake of clarity of the presentation let us assume that the payoffs of the agents can be the natural numbers only. Let us consider the coalition  $C_1 = \{a, b\}$  with the payoffs  $\phi^a = 3$ , and  $\phi^b = 2$ . If the coalition  $C_1$  is in the stable set, then the coalition  $C_2 = \{b, c\}$  with the payoffs  $\phi^b = 3$ , and  $\phi^c = 3$ , which dominates  $C_1$ , must not be in the stable set (otherwise it would contradict the internal stability requirement). Since  $C_2$  does not belong to the stable set, and it is dominated only by the coalition  $C_3 = \{a, c\}$  with the payoffs  $\phi^a = 2$ , and  $\phi^c = 3$ , we infer that  $C_3$  must belong to stable set. However,  $C_3$  is dominated by  $C_1$ , which leads to contradiction. By symmetry, we see that the stable set is empty.*

The same reasoning, as given in the example above, applies to the farsighted vN-M stable sets. The alternative definition in which the agents are conservative, the Coalitional Farsighted Conservative Stable Set was proposed by Diamantoudi and Xue [12]. Intuitively, in this definition the agents are willing to deviate only if every sequence starting from this deviation leads to a better outcome for them.

We believe that these two cases consider too extreme behavior of the agents. Nevertheless, we think that considering coalitional farsighted conservative stable sets in our game is a very appealing direction for the future work.

#### 4.3.4 Coalition-Proof Nash Equilibria

Another way of weakening the notion of the (rigorously) strongly winning coalition is to consider Coalition-Proof Nash Equilibria [7]. Intuitively, in the Coalitional-Proof Nash Equilibrium we first assume that all players are in a common room, where they can freely discuss their strategies. Then the agents, one by one, leave the room. Once the agent leaves the room, she cannot change her strategy. The agents that are left in the room are allowed to discuss and (cooperatively) change their strategies.

Unfortunately, these equilibria are not guaranteed to exist. This is what we expect since a Coalition-Proof Nash Equilibrium must be essentially a Nash Equilibrium. For the sake of completeness of the presentation below we show the appropriate example in which there is no Coalition-Proof Nash Equilibrium.

**Example 3.** *Consider the example from Proposition 14. There is a project with the budget  $v = 5$ ; and three identical agents  $a, b, c$  with minimal salaries  $\phi_i^{\min} > 2$ . Every coalition formed by any two agents is feasible. There is no Coalition-Proof Nash Equilibrium in this*

example (independently whether the salaries of the agents are natural or rational numbers). Indeed, consider any vector of payoffs  $\langle \phi^a, \phi^b, \phi^c \rangle$ . If  $\phi^a > 2$ , we infer that  $a$  forms a winning coalition with one of the agents  $b$ , or  $c$ . Without loss of generality we assume that  $\{a, b\}$  is the winning team. Thus,  $\phi^b < 3$  and  $\phi^c = 0$ . If we consider the subgame formed by the agents  $b$  and  $c$ , we see, however, that their payoff vector  $\langle \phi^b, \phi^c \rangle$  is Pareto-dominated by  $\langle 3, 2 \rangle$ . Now, let us consider the case when  $\phi^a < 2$ . One of the agents  $b$  and  $c$  needs to have payoff lower than 3 (w.l.o.g let us assume that this is the agent  $b$ ). But, if we consider the subgame formed by the agents  $a$  and  $b$ , their payoff vector  $\langle \phi^a, \phi^b \rangle$  is Pareto-dominated by  $\langle 2, 3 \rangle$ . Finally, let us assume that  $\phi^a = 2$ . We infer that one of the agents  $b$  and  $c$  gets zero payoff (let us assume that this is the agent  $b$ ). However, the payoff vector  $\langle \phi^a, \phi^b \rangle$  is Pareto-dominated by  $\langle 3, 2 \rangle$ .

#### 4.3.5 Graph Interpretations

Let us consider a directed multi-graph in which the vertices are the strategy profiles. Each pair of vertices can be connected with at most 2 edges, corresponding to implicit and explicit endangerment. Thus, the vertices  $v$  and  $u$  are connected by the edge corresponding to the implicit endangerment if and only if  $v$  is implicitly endangered by  $u$ . Analogously,  $v$  and  $u$  are connected by the edge corresponding to the explicit endangerment if and only if  $v$  is explicitly endangered by  $u$ .

Clearly, in such graph the strong Nash equilibria correspond to the sinks, the vertices with no outgoing edges. Also, the edges corresponding to explicit endangerment do not form cycles. Consequently, we can restrict our graph to the vertices that do not have outgoing edges corresponding to the explicit endangerment. We believe that every connected component in such restricted graph defines an interesting set of stable solutions, that we plan to analyze in our future work. For instance such defined set of stable solutions is always non-empty and its elements correspond to weakly winning coalitions, according to our definition from the previous section.

## 5 Mechanism Design

In this section we take a look at two mechanisms that a project issuer can apply to find a winning coalition: the first one sets the job's budget  $v$ ; the second one uses an first-price auction.

First, we show that if the client is allowed to change the value  $v$  there exists a simple mechanism ensuring the existence of the strongly winning coalition.

**Theorem 18.** *If there exists a feasible coalition, then there exists a budget  $v^*$  for which there exists a strongly winning coalition. The problem of finding such  $v^*$  can be solved in time  $O(\log v \cdot \text{ffc})$ , where  $\text{ffc}$  is the complexity of the problem FFC.*

*Proof.* Let  $v^*$  be the smallest value such that there exists a feasible coalition. We show that for  $v^*$  there exists a strongly winning coalition. Let  $\mathcal{C}^*$  be the most preferred (according to

the tie-breaking rule  $\prec$ ) feasible coalition for  $v^*$ . For the sake of contradiction let us assume that there exists a coalition  $\mathcal{C}'$  such that  $\mathcal{C}^*$  is strictly implicitly or explicitly endangered by  $\mathcal{C}'$ . Of course  $b_{\mathcal{C}'} \leq v^*$  (otherwise  $\mathcal{C}'$  would not be feasible). If  $\mathcal{C}^*$  is explicitly endangered by  $\mathcal{C}'$  ( $N_{\mathcal{C}^*} \cap N_{\mathcal{C}'} = \emptyset$ ), it means  $\mathcal{C}'$  is cheaper than  $\mathcal{C}^*$ ; and we get a contradiction with the definition of  $v^*$ . Otherwise ( $\mathcal{C}^*$  is strictly implicitly endangered by  $\mathcal{C}'$ ), let  $i \in N_{\mathcal{C}^*} \cap N_{\mathcal{C}'}$ . Now,  $i$  must get strictly better salary in  $\mathcal{C}'$  than in  $\mathcal{C}^*$ . Thus if we change the salary of  $i$  in the coalition  $\mathcal{C}'$  to  $\phi_{\mathcal{C}'}(i) = \phi_{\mathcal{C}^*}(i)$  we get a contradiction—a cheaper feasible coalition.

To find such a  $v^*$ , one has to run a binary search over  $v$ .  $\square$

In the second approach we use the first-price auction in which coalitions participate. In a standard first-price auction, an item's price starts from some minimal value (the least preferred outcome for the owner of the item). Bidders place bids for the current price. The asking price is gradually increased until there are no further bids; the last bidder wins the auction. Similarly, in our proposed auction, the auction starts from the original budget  $v$  (the least preferred outcome for the client); the asking price is gradually *decreased*. Coalitions place bids for the current asking price (as in the standard first-price auction, multiple bids for the same asking price are not allowed). The auction stops if there is no feasible coalition that can propose a lower bid than the current asking price. This leads to the concept of an auction-winning coalition.

**Definition 6.** *A coalition  $\mathcal{C}$  is auction-winning if and only if there is no feasible coalition  $\mathcal{C}'$  such that  $b_{\mathcal{C}'} < b_{\mathcal{C}}$  and for each agent  $i \in N_{\mathcal{C}} \cap N_{\mathcal{C}'}$  the agent gets better salary in  $\mathcal{C}'$ ,  $\phi_{\mathcal{C}'}(i) \geq \phi_{\mathcal{C}}(i)$ .*

**Proposition 19.** *The problem of checking whether a feasible coalition  $\mathcal{C}$  is auction-winning can be solved in time  $O(\text{ffc})$ . The problem of finding an auction-winning coalition can be solved in time  $O(v \cdot \text{ffc})$ ;  $\text{ffc}$  is the complexity of the problem FFC.*

*Proof.* To check whether a coalition  $\mathcal{C}$  is auction-winning one has to solve the problem of existence of the feasible coalition for the asking price:  $v = b_{\mathcal{C}} - 1$  (representing the next asking price in the first-price auction); and for each  $i \in N_{\mathcal{C}}$  set  $\phi_i^{\min} = \phi_{\mathcal{C}}(i)$  (these agents must get at least the same payoffs as in  $\mathcal{C}$ ). If no such coalition exists,  $\mathcal{C}$  is auction-winning.

To find an auction-winning coalition one can simply simulate the auction.  $\square$

Here, once again, we saw that these problems of finding auction-winning coalitions require solving the problem of finding a feasible coalition. We note that the procedure of finding an auction-winning coalition from Proposition 19 might be exponential with respect to the representation of  $v$ .

The summary of our results in general model is given in Table 1. We believe that the computational results favor the concept of the auction winning coalition (or the centralized model). First, the weakly winning coalition is guaranteed to exist. Second, the computational power needed to find an auction winning coalition seems much smaller in comparison with other concepts. In the centralized model, finding the winning coalition (when we already have the asking salaries of the agents) has also a straightforward reduction to FCFC.

## 6 Finding Feasible Coalitions in a Scheduling Model

In Sections 4 and 5 we show that many problems of finding the (weakly/strongly) winning coalitions or determining whether a given coalition is (weakly/strongly) winning required solving the subproblem of finding the feasible coalition. The general model (Section 2) assumed that given a coalition there is an oracle deciding whether there exists a feasible coalition.

By specifying an oracle, our results can be applied to two different problems known in the literature: the commodity auctions and the path auctions.

In the commodity auctions the project can be seen as the set of items  $I = \{i_1, i_2, \dots, i_q\}$  and each agent owns a certain subset of the items. The coalition is feasible if the agents have together all the items from  $I$ .

In the path auctions [26] we are given a graph  $G$  with two distinguished vertices: a source  $s$  and a target  $t$ . The agents correspond to the vertices in the graph. The coalition is feasible if the participating agents form a path from  $s$  to  $t$ .

In this section, we show a possible concrete instance of this model in which a project is a set of indivisible, independent tasks; and agents are processors who process these tasks with varying speeds.

### 6.1 The Scheduling Model

A project consists of a set  $\mathcal{T} = \{t_1, t_2, \dots, t_q\}$  of  $q$  independent tasks. The tasks can be processed sequentially or in parallel. The tasks are indivisible: a task must be processed on a single processor. Once started, a task cannot be interrupted. All tasks must be completed before  $d$ , the project's deadline.

Agents correspond to processors (in this section we use the words the agent and the processor interchangeably). Each agent has certain skills which are represented as the speed of executing the tasks. Thus, for each agent  $i$  we define the skill vector  $s_i = \langle s_{i,1}, s_{i,2}, \dots, s_{i,q} \rangle$  which has the following meaning: agent  $i$  is able to finish task  $t_j$  within  $s_{i,j}$  time units (with  $s_{i,j} = \infty$  when an agent is unable to finish the task). We assume that  $s_i$  is known (it can be well approximated from e.g. past behavior of the agent certified by clients in form of reviews). An agent can process only a single task at each time moment—if she wants to process more than one task, she must execute the tasks sequentially. We assume that only a single agent can work on a given task. This assumption is not as restrictive as it may appear; if the task  $t_i$  is large and can be processed by multiple agents in parallel, the project client will rather replace  $t_i$  by a number of smaller tasks.

For a coalition  $\mathcal{C}$  we define  $\Phi_{\mathcal{C}} : \mathcal{T} \rightarrow N_{\mathcal{C}}$  to be an assignment function (assigning tasks to agents). The assignment function  $\Phi_{\mathcal{C}}$  enables us to formalize the notion of a coalition completing the project before the deadline and also the total cost of the coalition. Specifically, a project is finished before the deadline  $d$  if and only if all the agents finish their assigned tasks before  $d$ ,  $\forall i \in N_{\mathcal{C}} \sum_{\ell: \Phi(t_{\ell})=i} s_{i,\ell} \leq d$ . In the hourly salary model, the cost of the coalition is equal to  $c_{\mathcal{C}} = \sum_{i \in N_{\mathcal{C}}} \phi_{\mathcal{C}}(i) \sum_{\ell: \Phi(t_{\ell})=i} s_{i,\ell}$ .

In the scheduling model we define the problem of finding a feasible coalition as follows.

**Problem 3** (FFCSM: Find Feasible Coalitions, Scheduling Model). *Let  $\mathcal{T}$  be the set of  $q$  tasks and  $N$  be the set of processors (or equivalently, agents). For each task  $t_j \in \mathcal{T}$  and each processor (agent)  $i \in N$  we define  $s_{i,j}$  as the processing time of  $t_j$  on  $i$ . Let  $\phi_i^{\min}$  be the cost of renting processor  $i$  (hiring agent  $i$ ). The budget of the project is  $v$  and the deadline is  $d$ . The FFCSM problem consists of selecting a subset of the processors  $N' \subseteq N$  and the assignment function  $\Phi : \mathcal{T} \rightarrow N'$  such that the budget is not exceeded ( $c_{N',\Phi} \leq B$ ) and the project's makespan does not exceed the deadline  $d$ .*

In the hourly salary model, the problem of finding the feasible coalition reduces to the problem of scheduling on unrelated processors with costs [29]. Specifically, Shmoys and Tardos [29] show a 2-approximation algorithm for approximating the makespan (the deadline  $d$  in our model).

**Problem 4** (FFCHS: Find Feasible Coalitions, Hourly Salary). *The instance of the problem is the same as in the FFCSM problem. In the FFCHS problem we additionally specify that the cost of the coalition  $c_{N',\Phi}$  is defined as  $cc = \sum_{i \in N_C} \phi_C(i) \sum_{\ell: \Phi(t_\ell)=i} s_{i,\ell}$ .*

The project salary model is a generalization of the problem of minimizing makespan on unrelated processors [31]. To the best of our knowledge this problem has not been stated before; thus we formally define it below.

**Problem 5** (FFCPS: Find Feasible Coalitions, Project Salary). *The instance of the problem is the same as in the FFCSM problem. In the FFCPS problem we additionally specify that the cost of the coalition  $c_{N',\Phi}$  is defined as  $c_{N',\Phi} = \sum_{i \in N'} \phi_i^{\min}$ .*

An easier problem to FFCPS, in which the goal is to optimize the assignment only (assuming that the processors are already selected) has a 2-approximation algorithm [31]. However, adding the notion of the budget usually significantly increases the complexity. We believe that the approximability of FFCPS is a very appealing problem.

## 6.2 FFCPS: Hardness Results

First, we show the NP-hardness of FFC-Scheduling in restricted special cases.

**Theorem 20.** *FFCPS and FFCHS are NP-hard even for two agents.*

*Proof.* The proof is by reduction from the partition problem. In the partition problem, we are given a set of integers  $\{n_j\}$ ; we ask whether there exists a partition of this set into two subsets  $S_1, S_2$ , such that  $\sum_{n_j \in S_1} n_j = \sum_{n_j \in S_2} n_j$ . To construct an instance of the feasible coalition problem, we construct a project that has a task for each  $n_j$ , an unlimited budget and a deadline  $d = 1/2 \sum n_j$ . We take two agents  $a$  and  $b$  with processing speeds  $s_{a,j} = s_{b,j} = n_j$  and unit costs:  $\phi_a^{\min} = \phi_b^{\min} = 1$ . A feasible coalition corresponds with partitioning numbers into two with equal sums.  $\square$

**Theorem 21.** *FFCPS is NP-hard even if the agents can be assigned no more than 3 tasks, if each agent has no more than 3 skills (for each  $j$  we have that  $\|\{i : s_{i,j} \neq \infty\}\| \leq 3$ ), if the deadline is constant, and if the minimal salaries of the agents are equal 1.*

*Proof.* The proof is by reduction from the exact set cover problem. In the exact set cover problem we are given a set of elements  $T = \{t_1, t_2, \dots, t_q\}$  and family  $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$  of 3-element subsets of  $T$ . We ask whether there exist  $\frac{q}{3}$  subsets from  $\mathcal{S}$  that cover all the elements from  $T$ . The exact set cover problem is NP-hard even if each member of  $T$  appears in at most 3 sets from  $\mathcal{S}$ .

We build an instance of the feasible coalition problem in the following way. There are  $q$  tasks and  $n$  agents; for each agent  $i$  and each task  $t_j$  we have that  $s_{i,j} = 1$  if and only if  $t_j \in S_i$ . Otherwise,  $s_{i,j} = \infty$ . The deadline  $d$  is equal to 3. The minimal salary of each agent is 1 and the budget  $v$  to  $\frac{q}{3}$ . It is easy to check that there exists a feasible coalition if and only if there exists a cover of  $T$  with  $\frac{q}{3}$  sets.  $\square$

**Theorem 22.** *FFCHS is NP-hard even if the agents can be assigned no more than 4 tasks, if each agent has no more than 4 skills (for each  $j$  we have that  $|\{i : s_{i,j} \neq \infty\}| \leq 4$ ), if the deadline is constant, and if the minimal salaries of the agents are equal 1.*

*Proof.* The proof is by reduction from the exact set cover problem. We are given a set of elements  $T = \{t_1, t_2, \dots, t_q\}$  and family  $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$  of 3-element subsets of  $T$ . We assume that each member of  $T$  appears in at most 3 sets from  $\mathcal{S}$ .

We build an instance  $I$  of the feasible coalition problem in the following way. There are  $q + n$  tasks and  $2n$  agents. The first  $q$  tasks  $t_1, t_2, \dots, t_q$  correspond to the elements in  $T$ . The next  $n$  tasks  $t_{q+1}, t_{q+2}, \dots, t_{q+n}$  are the dummy tasks needed by our construction. The first  $n$  agents  $1, 2, \dots, n$  correspond to the subsets from  $\mathcal{S}$  and the next  $n$  agents  $(n+1), (n+2), \dots, 2n$  are the dummy agents. The minimal salaries of all agents are equal to 1.

For each agent  $i$ ,  $i \leq n$  and each task  $t_j$ ,  $j \leq q$ , we set  $s_{i,j} = 2$  if and only if  $t_j \in S_i$ ; otherwise  $s_{i,j} = \infty$ . Also, for each agent  $i$ ,  $i \leq n$  and each task  $t_j$ ,  $j > q$  we set  $s_{i,j} = 5$  if and only if  $i = j - q$ ; otherwise  $s_{i,j} = \infty$ . For each agent  $i$ ,  $i > n$  and each task  $t_j$  we set  $s_{i,j} = 6$  if and only if  $i - n = j - q$ ; otherwise  $s_{i,j} = \infty$ . The deadline  $d$  is equal to 6 and the budget  $v$  is equal to  $v = \frac{7}{3}q + 5n$ . Clearly, each agent has no more than 4 skills and so, in any feasible solution, cannot be assigned more than 4 tasks.

We will show that the answer to the original instance of the exact set cover problem is “yes” if and only if there exists a feasible coalition in the our constructed instance  $I$ .

$\Leftarrow$  Let us assume there exists a feasible coalition  $\mathcal{C}$ . The cost of this coalition is at most equal to  $v = \frac{7}{3}q + 5n$ . Each non-dummy task (there are  $q$  such tasks) takes 2 time units, and thus implies the cost equal to 2. The dummy tasks can be assigned either to non-dummy agents (implying the cost 5) or to dummy agents (implying the cost 6). Thus, we infer that at most  $\frac{q}{3}$  dummy agents are assigned a task ( $2q + \frac{1}{3}q \cdot 6 + (n - \frac{1}{3}q) \cdot 5 = v$ ). As the result at least  $(n - \frac{q}{3})$  dummy tasks must be assigned to non-dummy agents. A non-dummy agent, who is assigned a dummy task cannot be assigned any other task (otherwise the completion time would exceed the deadline). Thus, at most  $\frac{q}{3}$  non-dummy agents can be assigned non-dummy tasks. The non-dummy tasks can be assigned only to non-dummy agents. We see the subsets corresponding to these non-dummy agents who are assigned non-dummy tasks form the solution to the initial exact set cover problem.

$\Rightarrow$  Let us assume that there exists the exact set cover in the initial problem. The agents corresponding to the subsets from the cover can be assigned tasks so that the deadline is not exceeded and the total cost of completing these tasks is equal to  $2q$ . The other  $(n - \frac{q}{3})$  non-dummy agents can be assigned one dummy task each. Finally, not-yet assigned dummy tasks can be assigned to dummy agents. The total cost of such assignment is equal to  $2q + (n - \frac{1}{3}q) \cdot 5 + \frac{1}{3}q \cdot 6 = v$ .

This completes the proof.  $\square$

Unfortunately, FFCPS is not approximable for makespan, for budget, and even for the combination of both parameters.

**Theorem 23.** *For any  $\alpha, \beta \geq 1$  there is no polynomial  $\alpha$ - $\beta$ -approximation algorithm for FFCPS that approximates makespan with the ratio  $\alpha$  and budget with the ratio  $\beta$ , unless  $P=NP$ . This result holds even if the costs of all processors are equal 1.*

*Proof.* For the sake of contradiction let us assume that there exists  $\alpha$ - $\beta$ -approximation algorithm  $A$ . We provide a reduction showing that  $A$  can be used as  $\beta$ -approximation algorithm for SETCOVER. This will however contradict the result of Feige [15]. Let  $I$  be an instance of SETCOVER, where  $T = \{t_1, t_2, \dots, t_q\}$  is the set of elements and  $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$  is the set of the subsets of  $T$ . We ask whether there exists  $K$  subsets from  $\mathcal{S}$  that together cover all elements from  $T$ .

From  $I$  we construct an instance of FFCPS in the following way. There are  $q$  tasks corresponding to  $q$  elements in  $I$ . There are  $n$  agents  $1, 2, \dots, n$  corresponding to the subsets in  $\mathcal{S}$ . The duration  $s_{i,j}$  of the task  $t_i$  when processed by the agent  $j$  is defined in the following way. If  $t_i \in S_j$  then  $s_{i,j} = 1$ . Otherwise,  $s_{i,j} = \alpha q + 1$ . The minimal salary of each agent is equal to 1 and the total budget is  $K$ . We show that if there exist  $K$  subsets from  $\mathcal{S}$  covering  $T$  then we can use  $A$  to find  $\beta K$  subsets covering  $T$ .

Let  $C$  denote the covering using  $K$  subsets. If we assign each task  $t_i$  to any agent  $j$  such that  $S_j \in C$  and  $t_i \in S_j$ , then the completion time of the tasks on each processor will be at most equal to  $q$ . In such case we will use only  $K$  processors. Thus  $A$  returns the solution with the makespan at most equal to  $\alpha q$  using at most  $\beta K$  processors. This, however, means that each task  $t_i$  is assigned to such agent  $j$  that  $t_i \in S_j$ . Thus, the subsets corresponding to the selected processors form the solution of  $I$ . Of course, there is at most  $\beta K$  such processors. This completes the proof.  $\square$

Theorems 20, 21, and 22 show that the problems FFCPS and FFCHS remain NP-hard even if various parameters are constant. Although Theorem 20 give us NP-hardness even for 2 agents, it is somehow not satisfactory as we used the fact that the deadline  $d$  can be very large. If the deadline is given in unary encoding, we can solve the case for 2 agents by dynamic programming. Thus, we asked the question, whether we can solve the problem efficiently for small number of agents, if the size of the input is given in unary encoding. We used the parameterized complexity theory [13] to approach this problem: we asked whether FFCPS and FFCHS belong to FPT for the parameter  $n$ —the number of the agents, provided the input is given in unary encoding.



Intuitively, FPT is a class of problems that can be solved in time  $f(n)m^{O(1)}$ , where  $m$  is the size of the input instance,  $n$  is a parameter, and  $f$  is a computable function. The class FPT is considered to be the class of the easy problems, while classes  $W[1] \subseteq W[2] \subseteq \dots$  are considered to be much harder. Unfortunately, FFCPS and FFCHS are also hard for the parameter  $n$ .

**Theorem 24.** *Consider the number of the agents as the parameter. FFCPS and FFCHS are  $W[1]$ -hard, even if the agents are the same, if for minimal salaries of the agents equal to 1, and if the size of the input is given in unary encoding.*

*Proof.* We show the reduction from Unary Bin Packing (which is  $W[1]$ -hard [19]). In the instance of the unary bin packing problem we are given a set  $T$  of  $q$  items  $T = \{t_1, t_2, \dots, t_q\}$  (the size of the item  $t_i$  is equal to  $s_i$ ) and a set  $N$  of  $n$  bins, each having a capacity  $d$ . We ask whether it is possible to pack all the items to the bins.

From this instance we can construct the instance of FFCPS (or FFCHS) in the following way. Here  $T$  will be the set of tasks,  $N$  will be the set of agents. The minimal salaries of the agents are equal to 1; the speed of processing the task  $t_j$  by the agent  $i$  is equal to  $s_{i,j} = s_j$ . In FFCPS we set the total budget  $v$  to be equal to  $n$ . In FFCHS we set  $v$  to  $\sum_{t_i \in T} s_i$ . Of course, there exists a feasible schedule if and only if there exists a feasible bin-packing.  $\square$

### 6.3 Integer Programming Formulation

In the hourly salary model, Shmoys and Tardos [29] show an integer programming formulation. In this subsection we state the FFCPS problem as an integer programming problem.

$$\text{minimize } d \tag{8}$$

$$\text{subject to } \sum_{i \in N} a_i \phi_i^{\min} \leq v \tag{9}$$

$$x_{i,j} \leq a_i, \quad i \in N \tag{10}$$

$$\sum_{t_j \in T} x_{i,j} s_{i,j} \leq d, \quad i \in N \leq d \tag{11}$$

$$x_{i,j} \in \{0, 1\}, \quad i \in N; t_j \in T \tag{12}$$

$$a_i \in \{0, 1\}, \quad i \in N \tag{13}$$

In the above formulation, a binary variable  $a_i$  denotes whether agent  $i$  is a part of the solution (is assigned some tasks, Equation 13). A binary variable  $x_{i,j}$  is equal to 1 if and only if the task  $t_j$  is assigned to the agent  $i$  (Equation 12). We minimize the makespan  $d$  (Equation 8), which is the maximal completion time of the tasks over all the agents (Inequality 11). We cannot exceed the budget  $v$  (Inequality 9), and the tasks can be assigned only to the selected agents (Inequality 10).

Table 1: The summary of the results in general model. The column “Existence” contains the information whether a coalition/equilibrium always exists. The column “Checking” contains the complexity of checking whether a given coalition satisfies the definition corresponding to the row. The column “Finding” contains the complexity of finding a coalition/equilibrium ( $ffc$  and  $fcfc$  are the complexities of the problems FFC and FCFC, respectively). The values marked as (\*) are valid only in the project salary model. The values marked as (+) are valid only in the hourly salary model. The values marked as (-) are valid only if the salaries of the agents can be rational numbers.

		Existence	Checking	Finding
Decentr.	rigorously strongly winning c.	Not always	$O(n^2 \cdot fcfc)$	$O(n^5 \log(nv)fcfc)$ (*) (-)
	strongly winning coalition	Not always	open problem	
	weakly winning coalition	Always	$O(n^5 \log(nv)fcfc)$ (*) (-)	
	auction winning coalition	Always	$O(ffc)$	$O(v \cdot ffc)$
Central.	winning coalition (having asking salaries)	N/A	$O(fcfc)$	
	Strong Nash Equilibrium	Always (*) Not always (+)	$O(fcfc)$	$O(n^3 \log(nv)fcfc)$ (*) (-)

## 7 Conclusions

In this paper we present a new class of coalitional games that model cooperation and competition for the employment in a complex project. We believe that this is an interesting setting that relates to other natural problems, like coalition formation, coalitional auctions, auctions for sharable items, etc. We consider two models of the organization of the market. First, the winning coalition is selected by a central mechanism; the agents are strategic about the salaries they ask. Second, the coalition formation process is decentralized—the already-formed coalitions bid for the project, thus the agents are strategic both about the salaries and their cooperation partners.

We propose the concepts of stability in each model. These concepts are of interest both to the agents and to the client. The client gains an insight into agents’ strategies and can thus establish a relation between the cost of organizing the market and the cost of the winning coalition. The agents can optimize their strategies according to their beliefs (an agent can ask e.g. whether she can increase her asking salary and still participate in the winning coalition). In the centralized model we show that the Strong Nash Equilibrium always exists. In the decentralized model the SNE may not exist, but we prove the existence of a weakly winning coalition. We show how to reduce the problem of finding a winning coalition to the problem of finding a feasible one. Our results are summerized in Table 1. Finally, to show that the abstract model can be applied in practice, we present a concrete model in which the project is represented as a set of independent tasks and the agents have certain skills (expressed as the processing speeds). We prove the hardness of the problems in restricted cases.

There are many natural open questions. The first interesting direction is to consider other solution concepts to the decentralized variant of our model. As pointed out in Section 4.3, considering Coalitional Farsighted Conservative Stable Set and the concepts inspired by the graph interpretations is a promising avenue. Analyzing the complexity of computing these equilibria, as well as analyzing their game-theoretic properties are both open questions. Second, it is natural to consider other auctions in the decentralized setting. In particular it is an open question whether there exists a truthful mechanism in this setting. This question is especially appealing in the light of the vast literature on designing the truthful mechanisms for the centralized setting. Third, it is interesting to analyze how incomplete knowledge of the agents and the inconsistencies of their beliefs affect the equilibria. How should an agent play when she is aware of these inconsistencies? These are just ones of many interesting directions to continue the work on the auctions for the complex projects.

## References

- [1] Independent contractors: How many? (australia). [www.independentcontractors.net.au/Research/How-Many/independent-contractors-how-many](http://www.independentcontractors.net.au/Research/How-Many/independent-contractors-how-many).
- [2] N. Andersson. *The Single And Multi Project Approach To Planning And Scheduling*. CBS, 2008.
- [3] A. Archer and É. Tardos. Frugal path mechanisms. *ACM Transactions on Algorithms*, 3(1):3:1–3:22, February 2007.
- [4] R. J. Aumann. Acceptable points in general cooperative N-person games. In *Contribution to the theory of game IV, Annals of Mathematical Study 40*, pages 287–324. University Press, 1959.
- [5] David Baccarini. The concept of project complexitya review. *International Journal of Project Management*, 14(4):201–204, 1996.
- [6] Y. Bachrach, D. C. Parkes, and J. S. Rosenschein. Computing cooperative solution concepts in coalitional skill games. *Artificial Intelligence*, 204(0):1–21, 2013.
- [7] B. D. Bernheim, B. Peleg, and M. D. Whinston. Coalition-proof nash equilibria i. concepts. *Journal of Economic Theory*, 42(1):1–12, 1987.
- [8] P. Charrel and D. Galarreta. *Project Management and Risk Management in Complex Projects*. Springer, November 2010.
- [9] N. Chen, E. Elkind, N. Gravin, and F. Petrov. Frugal mechanism design via spectral techniques. In *Proceedings of FOCS-2010*, pages 755–764, 2010.

- [10] N. Chen and A. R. Karlin. Cheap labor can be expensive. In *Proceedings of SODA-2007*, pages 707–715, 2007.
- [11] M. S. Chwe. Farsighted coalitional stability. *Journal of Economic Theory*, 63:299–325, 1994.
- [12] E. Diamantoudi and L. Xue. Farsighted stability in hedonic games. *Social Choice and Welfare*, 21(1):39–61, 2003.
- [13] R. Downey and M. Fellows. *Parameterized Complexity*. Springer-Verlag, 1999.
- [14] D. J. Edwards. Accident trends involving construction plant: An exploratory analysis. *Journal of Construction Research*, 04(02):161–173, 2003.
- [15] U. Feige. A threshold of  $\ln n$  for approximating set cover. *J. ACM*, 45(4):634–652, 1998.
- [16] R. M. Freund and J. R. Vera. Equivalence of convex problem geometry and computational complexity in the separation oracle model. *Mathematics of Operations Research*, 34(4):869–879, 2009.
- [17] R. Garg, V. Kumar, A. Rudra, and A. Verma. Coalitional games on graphs: Core structure, substitutes and frugality. In *Proceedings of ACM-EC-2003*, pages 248–249, 2003.
- [18] A. Iwasaki, D. Kempe, Y. Saito, M. Salek, and M. Yokoo. False-name-proof mechanisms for hiring a team. In *Proceedings of WINE-2007*, pages 245–256, 2007.
- [19] K. Jansen, S. Kratsch, D. Marx, and I. Schlotter. Bin packing with fixed number of bins revisited. In *Proceedings of SWAT-2010*, pages 260–272, 2010.
- [20] A. R. Karlin, D. Kempe, and T. Tamir. Beyond vcg: Frugality of truthful mechanisms. In *Proceedings of FOCS-2005*, pages 615–626, 2005.
- [21] L. G. Khachiyan. A polynomial algorithm in linear programming. *Doklady Akademiia Nauk SSSR*, 244, 1979.
- [22] V. Krishna. *Auction Theory*. Academic Press, March 2002.
- [23] Antony McCabe. *Frugality in Set-System Auctions*. PhD thesis, University of Liverpool, 2012.
- [24] J. Moules. Number of freelancers on the increase. *Financial Times*, 2011.
- [25] J. F. Nash. Equilibrium points in  $n$ -person games. In *Proceedings of the National Academy of Sciences of the United States of America*, volume 36, pages 48–49, 1950.
- [26] N. Nisan and A. Ronen. Algorithmic mechanism design (extended abstract). In *Proceedings of STOC-1999*, pages 129–140, 1999.

- [27] J. M. Osborne and A. Rubinstein. *A Course in Game Theory*, volume 1 of *MIT Press Books*. The MIT Press, 1994.
- [28] T. Rahwan, T. P. Michalak, M. Wooldridge, and N. R. Jennings. Anytime coalition structure generation in multi-agent systems with positive or negative externalities. *Artificial Intelligence*, 186:95–122, 2012.
- [29] D. B. Shmoys and É. Tardos. Scheduling unrelated machines with costs. In *Proceedings of SODA-1993*, pages 448–454, 1993.
- [30] K. Talwar. The price of truth: Frugality in truthful mechanisms. In *Proceedings of STACS-2003*, pages 608–619, 2003.
- [31] Vijay V. Vazirani. *Approximation algorithms*. Springer-Verlag New York, Inc., New York, NY, USA, 2001.
- [32] K. D. Walsh, A. Sawhney, and H. H. Bashford. Cycle-time contributions of hyper-specialization and time-gating strategies in us residential construction. In *Proceedings of 11th Annual Conference on Lean Construction-2003*, pages 390–397, 2003.
- [33] T. M. Williams. The need for new paradigms for complex projects. *International Journal of Project Management*, 17(5):269 – 273, 1999.
- [34] M. Wooldridge and P. E. Dunne. On the computational complexity of coalitional resource games. *Artificial Intelligence*, 170(10):835–871, July 2006.